Lecture 14 (2/4/22) - 15 (2/11/22)

Recall. • For $f \in \mathcal{M}(G)$, $z \in G$,

$$\mu(f)(z) = \begin{cases} \dfrac{2|f'(z)|}{1+|f(z)|^2}, & z \text{ not a pole} \\ \lim_{\zeta \to z} \mu(f)(\zeta), & z \text{ a pole.} \end{cases}$$

and $\mu(f) \in C(G, \mathbb{C})$.

Thm 1. Let $\mathcal{F}$ be a family in $\mathcal{M}(G)$. Then $\mathcal{F}$ normal $\Leftrightarrow$ $\mu(\mathcal{F})$ is locally bdd.

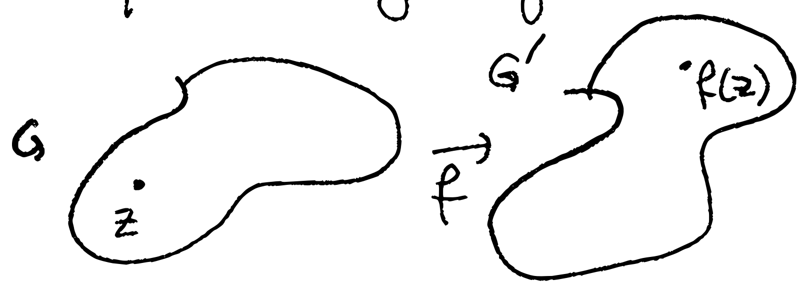
Pf. "$\Rightarrow$" HW. "$\Leftarrow$". Finish pf from Lecture 13 notes. ∎

Conformal Equivalence.

Recall. A conformal map f of $G \subseteq \mathbb{C}$ onto $G' \subseteq \mathbb{C}$ (both open) is an analytic function that gives a 1:1 map of G onto G'. A conformal map $f: G \to G'$ has an inverse (also conformal) $f^{-1}: G' \to G$.

The domains G, G' are said to be <u>conformally equivalent</u> $(G \cong G')$ when a conformal map $f : G \xrightarrow{\cong} G'$ exists.

<u>Rem.</u> A conformal equivalence $f : G \xrightarrow{\cong} G'$ induces an isomorphisms
$H(G') \xrightarrow{\cong} H(G)$, $\mathcal{M}(G') \xrightarrow{\cong} \mathcal{M}(G)$
by composition $g \to g \circ f$



<u>Philosophy</u>: Anything "complex analytic" in G' transfers to G.

Therefore, it is useful to find model domains for $G \subseteq \mathbb{C}$.

Lecture 15 (2/11/22).

Riemann Mapping Thm. Let $G \subseteq \mathbb{C}$ be a simply connected region, $G \neq \mathbb{C}$. For any $a \in G$ $\exists$ unique conformal map $f: G \to \mathbb{D} = \{z : |z| < 1\}$ s.t. $f(a) = 0$, $f'(a) > 0$.

Rem. • There cannot exist such f when $G = \mathbb{C}$ by Liouville's Thm.

• Uniqueness is clear. If $f, \tilde{f}$ are both such conformal maps, then $\varphi = f \circ \tilde{f}^{-1}$ is an automorphism of $\mathbb{D}$. Moreover, $\varphi(0) = 0$, $\varphi'(0) > 0$. By classification of $\mathrm{Aut}(\mathbb{D})$, we see that $\varphi(z) = z$ is the only possibility, which establishes uniqueness.

Pf. The proof is by a normal family argument. The idea (common in math) is to set up an extremal problem.

What prop's of f do we need? Well,

(i) $f(a)=0$, $f'(a)>0$

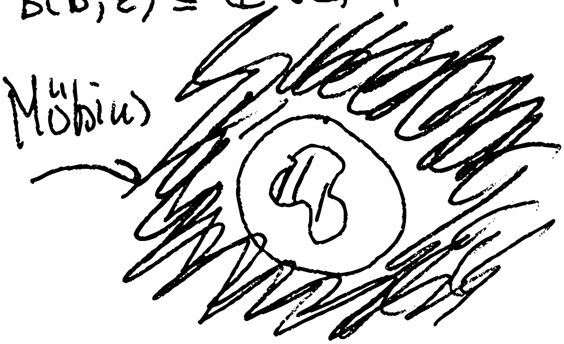
(ii) f is 1:1.

(iii) $f(G)=\mathbb{D}$.

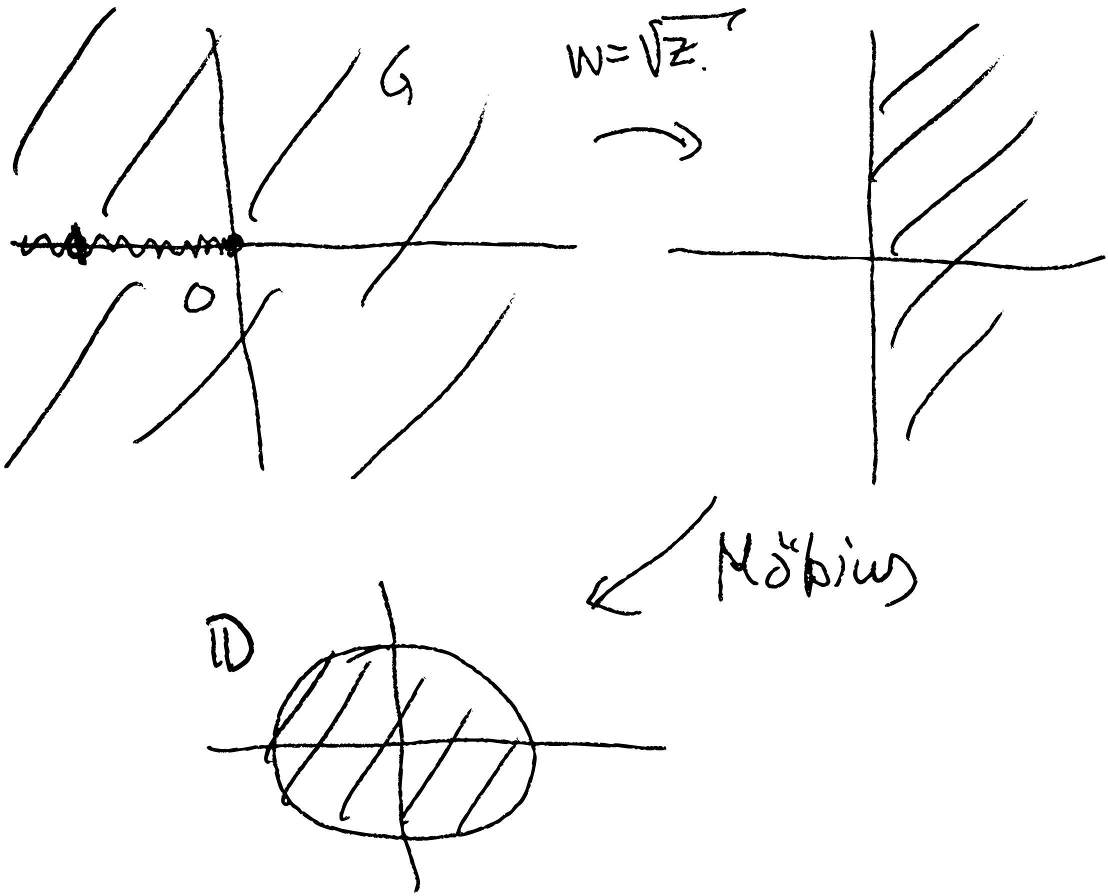
Relax (iii) to $\boxed{(iii)' \ f(G)\subseteq\mathbb{D}}$ so that we can more easily find candidates and then find the one that maximizes $f'(a)$. Show this satisfies (iii).

Thus, let $\mathcal{F}=\{f\in H(G): f$ satisfies (i), (ii), and (iii)' $f(G)\subseteq\mathbb{D}\}$. We proceed in steps:

(1) $\mathcal{F}\neq\emptyset$. Note that if G is bounded, this is trivial. When G is not bdd, this is less trivial and uses $G\neq\mathbb{C}$. So let $b\in\mathbb{C}\setminus G$. If $\exists B(b,\varepsilon)\subseteq\mathbb{C}\setminus G$ then it is simple:

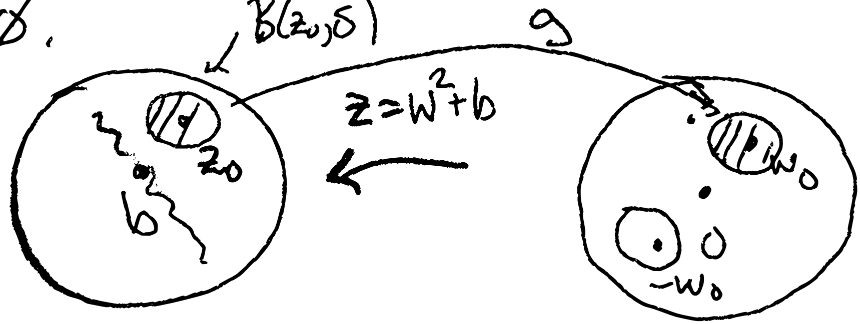


If $\nexists\, B(b,\varepsilon) \subseteq \mathbb{C} \setminus G$, we need a trick.
Here's the idea in an example



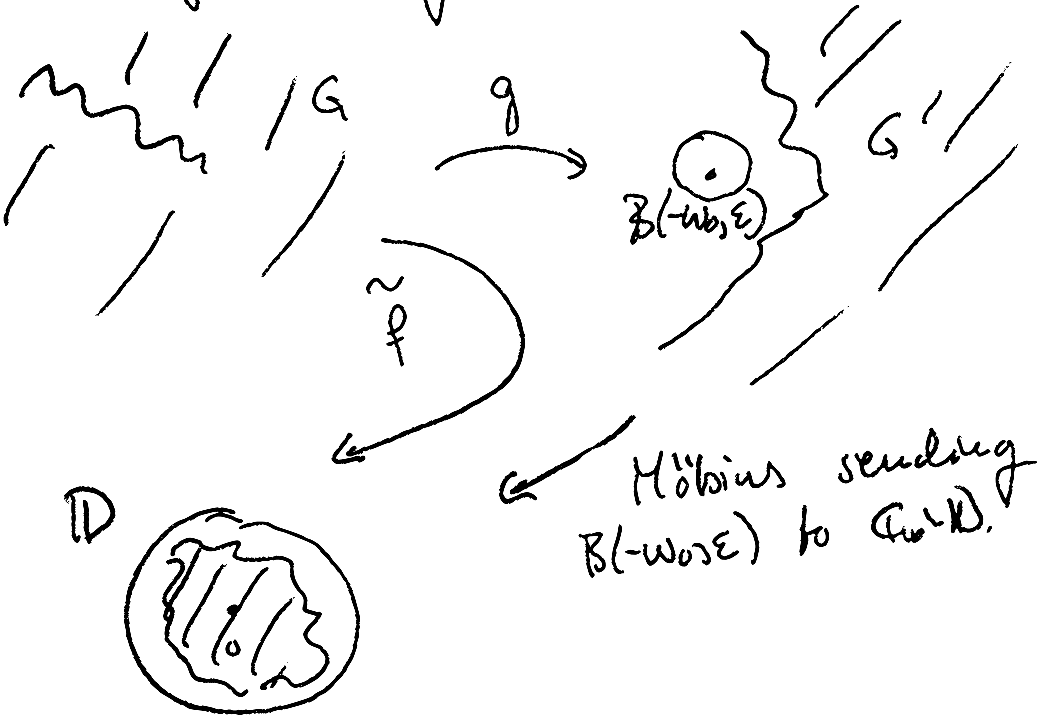
The point is now that when G is simply connected, since $b \notin G$, there is a branch g of $\sqrt{z-b}$ $\left(= e^{\frac{1}{2}\log(z-b)}\right)$.
We will "open up" $\mathbb{C} \setminus G$ near b by using g.

For $z_0 \in G$, we let $w_0 = g(z_0)$. Then $w_0^2 = z_0 - b$ and there are two solutions $w_0 = g(z_0) \neq 0$ and $-w_0$. We claim that there is $\varepsilon > 0$ s.t. $B(-w_0, \varepsilon) \cap g(G) = \emptyset$.



Let $\varepsilon' > 0$ s.t. $B(w_0, \varepsilon') \cap B(-w_0, \varepsilon') = \emptyset$. Let $\delta > 0$ s.t. $B(z_0, \delta) \subseteq G$ and $g(B(z_0, \delta)) \subseteq B(w_0, \varepsilon')$. Now, let $\varepsilon < \varepsilon'$ be s.t. $w \to z = w^2 + b$ sends $B(w_0, \varepsilon)$ and $B(-w_0, \varepsilon)$ into $B(z_0, \delta)$. If $w \in B(-w_0, \varepsilon)$ and $\exists\, z$ s.t. $g(z) = w$, then $z = w^2 + b$ $\Rightarrow z \in B(z_0, \delta)$ but $g(B(z_0, \delta)) \subseteq B(w_0, \varepsilon')$ and $w \notin B(w_0, \varepsilon')$. Thus, $B(-w_0, \varepsilon) \cap g(G) = \emptyset$ as claimed.

g is a conformal map $G \to G' = g(G)$ (since g is clearly 1:1).



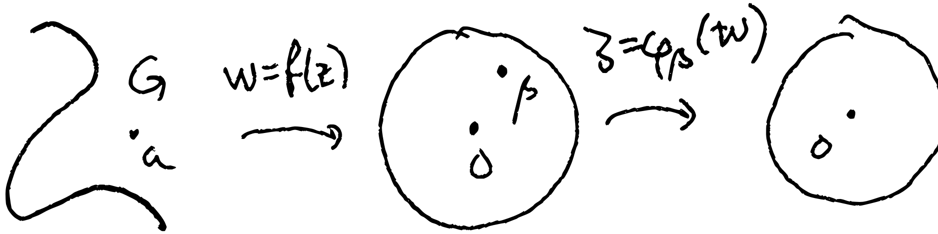
$\tilde{f}$ is a conformal map $G \to \mathbb{D}$ (not nec. surjective). After composing also with $e^{i\theta}\varphi_{\tilde{a}} \in \mathrm{Aut}(\mathbb{D})$, $\tilde{a} = \tilde{f}(a)$ and $e^{i\theta}\tilde{f}'(a) > 0$, we obtain $\mathcal{F} \neq \emptyset$, as claimed.

(2). Let $\alpha = \sup\limits_{\mathcal{F}} f'(a)$ and let $\{f_n\}_{n=1}^{\infty}$ be seq. in $\mathcal{F}$ s.t. $f_n'(a) \nearrow \alpha$. By Montel, $\exists$ subseq. $f_{n_k} \to f$ in $H(G)$.

We claim $f \in \mathcal{F}$. Well, clearly f satisfies (i). Next, recall that if $f_n \to f$ in $H(G)$ and f_n are 1:1, then either f is 1:1 or f is constant (when G is a region). This was a HW problem from VII.2. But f cannot be constant since $f'(a) = \alpha > 0$.

Finally, since $f_n(G) \subseteq \mathbb{D} \Rightarrow f(G) \subseteq \overline{\mathbb{D}}$. But since f is not constant, $f(G)$ is open $\Rightarrow f(G) \subseteq \mathbb{D}$, i.e. (iii') so $f \in \mathcal{F}$ as claimed.

③ The last step is to show that $f(G)=\mathbb{D}$. Again, we use the existence of $\sqrt{g}$ for $g\neq 0$ in G. Suppose $\beta\in\mathbb{D}\setminus f(G)$. Consider $h: G\to\mathbb{D}$ given by



$h(z)=\sqrt{(\varphi_\beta\circ f)(z)}$. Note: h is 1:1, $h(G)\subseteq\mathbb{D}$, but $h\notin\mathcal{F}$. Thus, we correct by composing with $c\cdot\varphi_{h(a)}$:

$$g(z)=\frac{|h'(a)|}{h'(a)}\,\frac{h(z)-h(a)}{1-\overline{h(a)}\,h(z)}$$

Now, $g\in\mathcal{F}$ if $g'(a)>0$.

We compute: $g = c\,\varphi_\gamma \circ h$ where

$c = \dfrac{|h'(a)|}{h'(a)}$, $h = \sqrt{\varphi_\beta \circ f}$, $\gamma = h(a) = \sqrt{-\beta}$

$$g'(a) = c\,\varphi_\gamma'(\gamma)\cdot h'(a) = |h'(a)|\,\frac{1}{1-|\gamma|^2}.$$

$$h'(a) = \frac{1}{2h(a)}\,\varphi_\beta'(0)\,f'(a)$$

$$= \frac{1}{2\gamma}(1-|\beta|^2)\,\alpha \quad \Rightarrow$$

$$g'(a) = \frac{1}{2|\gamma|}\,\frac{(1-|\beta|^2)}{(1-|\gamma|^2)}\,\alpha.$$

Now, $-\beta = \gamma^2 \Rightarrow |\beta| = |\gamma|^2 \Rightarrow$

$$g'(a) = \frac{1+|\beta|}{2\sqrt{|\beta|}}\,\alpha$$

Since $0 < (1-\sqrt{|\beta|})^2 = 1+|\beta|-2\sqrt{|\beta|} \Rightarrow$

$\dfrac{1+|\beta|}{2\sqrt{|\beta|}} > 1 \Rightarrow g'(a) > \alpha$. But then

$g \in \mathcal{F}$ and $g'(a) > \sup\limits_{\mathcal{F}} |f'(a)|$. $\times \Rightarrow f(G) = \mathbb{D}$.

$\boxtimes$